

Decay properties of smooth axially symmetric D-solutions to the steady Navier-Stokes equations

Shangkun Weng*

November 4, 2015

Abstract

We investigate the decay properties of smooth axially symmetric D-solutions to the steady Navier-Stokes equations. The achievements of this paper are two folds. One is improved decay rates of u_θ and $\nabla \mathbf{u}$, especially we show that $|u_\theta(r, z)| \leq c \left(\frac{\log r}{r} \right)^{\frac{1}{2}}$ for any smooth axially symmetric D-solutions to the Navier-Stokes equations. These improvement are based on improved weighted estimates of ω_θ , integral representations of \mathbf{u} in terms of $\omega = \text{curl } \mathbf{u}$ and A_p weight for singular integral operators, which yields good decay estimates for $(\nabla u_r, \nabla u_z)$ and (ω_r, ω_z) , where $\omega = \omega_r \mathbf{e}_r + \omega_\theta \mathbf{e}_\theta + \omega_z \mathbf{e}_z$. Another is the first decay rate estimates in the O_z -direction for smooth axially symmetric flows without swirl. We do not need any small assumptions on the forcing term.

Mathematics Subject Classifications 2010: Primary 76D05; Secondary 35Q35.

Key words: Navier-Stokes, axially symmetric, decay, A_p weight.

1 Introduction and main results

In this paper, we will investigate the decay properties of the smooth axially symmetric solutions to the steady Navier-Stokes equations

$$\begin{cases} (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \Delta \mathbf{u} = \mathbf{f}, & \forall \mathbf{x} \in \mathbb{R}^3 \\ \text{div } \mathbf{u} = 0, \\ \lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty = 0 \end{cases} \quad (1.1)$$

with finite Dirichlet integral

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}(\mathbf{x})|^2 d\mathbf{x} < +\infty. \quad (1.2)$$

Here \mathbf{u} , p and \mathbf{f} denote the fluid velocity, the pressure and the body force. \mathbf{u}_∞ is a constant vector and we also assume the viscosity to be 1 for simplicity. One can also consider the same problem in the exterior domains $\Omega \subset \mathbb{R}^3$ with the no-slip boundary condition on $\partial\Omega$, where the complement of Ω is a compact axially symmetric domain. For simplicity, we only consider the whole space case and $\mathbf{u}_\infty = 0$ in this paper, some of our results can also be extended to the exterior domain case.

*Pohang Mathematics Institute, Pohang University of Science and Technology. Pohang, Gyungbuk, 790-784, Republic of Korea. Email: skwengmath@gmail.com.

The fundamental contribution to the existence of weak solutions to the stationary Navier-Stokes equations is due to Leray [21], where he constructed the weak solution of (1.1) with no-slip boundary conditions and constant velocity at infinity. Leray's solution has finite Dirichlet integral, and is usually refereed as D -solution. Ladyzhenskaya [20] and Fujita [13] also considered the nonhomogeneous boundary conditions case. It was easy to show that the D -solutions are smooth provided that data are smooth. In [10, 11], Finn showed that any D -solution in three dimensional exterior domain converged uniformly pointwise to the prescribed vector \mathbf{u}_∞ at infinity and, moreover, in the case $\mathbf{u}_\infty \neq 0$, he showed that if $|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty| \leq C|\mathbf{x}|^{-\alpha}$ for some $\alpha > \frac{1}{2}$ as $\mathbf{x} \rightarrow \infty$, then $|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty| \leq \bar{C}|\mathbf{x}|^{-1}$ as $\mathbf{x} \rightarrow \infty$. Finn also suggested a class of physical reasonable (PR) solutions to (1.1) in the three-dimensional exterior domain satisfying $\mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ if $\mathbf{u}_\infty = 0$ or $\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty = O(|\mathbf{x}|^{-\frac{1}{2}-\epsilon})$ for some $\epsilon > 0$, if $\mathbf{u}_\infty \neq 0$. Finn [12] then established the existence and uniqueness of a physically reasonable solution in a three dimensional exterior domain when the data are small enough. It is easy to show a PR-solution is a D -solution. However, whether the converse implication holds true has remained open for long times. In the case of $\mathbf{u}_\infty \neq 0$, Babenko [1] showed that every D -solution is a PR-solution if the force is of bounded support. Galdi [15] also proved the same result for $\mathbf{u}_\infty = 0$, under the assumption that \mathbf{u} obeys the "energy inequality" and the viscosity is sufficiently large. In [23], the authors established the existence and uniqueness of solution to (1.1) with the same decay rate as that of the fundamental solution of the Stokes problem, under some smallness assumptions on the data. For the investigation of the asymptotic profile of (1.1) with $\mathbf{u}_\infty = 0$, one can refer to [6, 18, 22]. For more information about the recent results about these problems, one can refer to [8, 9, 14].

From the above introduction, we see that the theory about the decay properties of D -solutions to (1.1) is quite incomplete in the case of large forcing term. In this article, we will investigate the D -solution \mathbf{u} with additional axially symmetric property to simplify this problem. On the other hand, this paper can be regarded as a continuation of my previous study [5] with Prof. Chae, where we established some interesting Liouville type theorems for smooth axially symmetric D -solutions. From [5], we can see that there is a close relation between the decay properties of D -solution to (1.1) with the famous open problem of the triviality of D -solution to (1.1) with $\mathbf{f} \equiv 0$ and $\mathbf{u}_\infty = 0$. One can also refer to [3, 4, 5, 14, 19] for more recent results about this triviality problem. Now we introduce the mathematical setup of our problem. More precisely, we introduce the cylindrical coordinate

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3.$$

We denote $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ the standard basis vectors in the cylindrical coordinate:

$$\mathbf{e}_r = (\cos \theta, \sin \theta, 0), \quad \mathbf{e}_\theta = (-\sin \theta, \cos \theta, 0), \quad \mathbf{e}_z = (0, 0, 1).$$

A function f is said to be *axially symmetric* if it does not depend on θ . A vector-valued function $\mathbf{u} = (u_r, u_\theta, u_z)$ is called *axially symmetric* if u_r, u_θ and u_z do not depend on θ . A vector-valued function $\mathbf{u} = (u_r, u_\theta, u_z)$ is called *axially symmetric with no swirl* if $u_\theta = 0$ while u_r and u_z do not depend on θ .

Assume that $\mathbf{u}(\mathbf{x}) = u_r(r, z)\mathbf{e}_r + u_\theta(r, z)\mathbf{e}_\theta + u_z(r, z)\mathbf{e}_z$ is a smooth D -solution to (1.1). The corresponding asymmetric steady Navier-Stokes equations read as follows.

$$\begin{cases} (u_r \partial_r + u_z \partial_z)u_r - \frac{u_\theta^2}{r} + \partial_r p = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_r + f_r, \\ (u_r \partial_r + u_z \partial_z)u_\theta + \frac{u_r u_\theta}{r} = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u_\theta + f_\theta, \\ (u_r \partial_r + u_z \partial_z)u_z + \partial_z p = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) u_z + f_z, \\ \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0. \end{cases} \quad (1.3)$$

Define the vorticity $\omega(\mathbf{x}) = \text{curl } \mathbf{u}(\mathbf{x}) = \omega_r(r, z)\mathbf{e}_r + \omega_\theta(r, z)\mathbf{e}_\theta + \omega_z(r, z)\mathbf{e}_z$, where

$$\omega_r = -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \frac{1}{r}\partial_r(ru_\theta).$$

The equations satisfied by ω_r , ω_θ and ω_z are listed as follows.

$$(u_r\partial_r + u_z\partial_z)\omega_r - (\omega_r\partial_r + \omega_z\partial_z)u_r = \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right)\omega_r - \partial_z f_\theta, \quad (1.4)$$

$$(u_r\partial_r + u_z\partial_z)\omega_\theta - \frac{u_r\omega_\theta}{r} - \frac{1}{r}\partial_z(u_\theta^2) = \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right)\omega_\theta + \partial_z f_r - \partial_r f_z, \quad (1.5)$$

$$(u_r\partial_r + u_z\partial_z)\omega_z - (\omega_r\partial_r + \omega_z\partial_z)u_z = \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)\omega_z + \frac{1}{r}\partial_r(rf_\theta). \quad (1.6)$$

For the investigation of the decay properties of D-solutions to the exterior stationary Navier-Stokes equations (1.1), one should trace back to the important papers by Gilbarg and Weinberger [16, 17]. For the two dimensional exterior domain, Gilbarg and Weinberger [16] showed that the weak solution constructed by Leray was bounded and converged to a limit \mathbf{u}_0 in a mean square sense, while the pressure converged pointwise. In [17], they further investigated any weak solutions to (1.1) with finite Dirichlet integral, and found that the weak solution \mathbf{u} may not be bounded, but it grew more slowly than $(\log r)^{\frac{1}{2}}$. The pressure has a finite limit at infinity and the velocity either had a limit in the mean or $\int_0^{2\pi} |\mathbf{u}(r, \theta)|^2 d\theta \rightarrow \infty$ as $r \rightarrow \infty$. The vorticity $\omega = \partial_{x_2} u_1 - \partial_{x_1} u_2$ decayed more rapidly than $r^{-\frac{3}{4}}(\log r)^{\frac{1}{8}}$ and the first derivatives of the velocity decayed more rapidly than $r^{-\frac{3}{4}}(\log r)^{\frac{9}{8}}$ at infinity.

Inspired by [16] and [17], Choe and Jin [7] first obtained the following decay rates for smooth axially symmetric solutions to (1.1): Let Ω be an exterior domain, suppose $\mathbf{f} \in H^1(\Omega)$ be an axially symmetric vector field with

$$\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{f}\|_{H^1(\Omega)} + \left\| \left(\frac{r}{\log r} \right)^{1/2} f_\theta \right\|_{L^2(\Omega)} + \|r\partial_z f_r\|_{L^2(\Omega)} + \|r\partial_r f_z\|_{L^2(\Omega)} \leq M \quad (1.7)$$

for some constant M . Then the axially symmetric solution (\mathbf{u}, p) with finite Dirichlet integral satisfied

$$|u_r(r, z)| + |u_z(r, z)| \leq c(M) \left(\frac{\log r}{r} \right)^{\frac{1}{2}}, \quad (1.8)$$

$$|u_\theta(r, z)| \leq c(M) \frac{(\log r)^{1/8}}{r^{3/8}}, \quad (1.9)$$

$$|\omega_\theta(r, z)| \leq \frac{c(M)}{r^{\frac{7}{8}}} \quad (1.10)$$

for large r .

Based on these results, we can obtain better improved decay rate estimates for u_θ , ω and $\nabla \mathbf{u}$. We first obtain some weighted and decay estimates of ∇u_r and ∇u_z by A_p weight method, since ∇u_r and ∇u_z can be expressed as singular integral operators of ω_θ . These yield weighted energy and decay estimates of ω_r and ω_z by using the equations of ω_r and ω_z . Finally, we use the integral formula of u_θ in terms of ω_r and ω_z to improve the decay rates. Our first main result is stated as follows.

Theorem 1.1. Suppose $\mathbf{f} \in H^1(\mathbb{R}^3)$ be an axially symmetric vector field with

$$\|\mathbf{f}\|_{H^{-1}(\mathbb{R}^3)} + \|\mathbf{f}\|_{H^1(\mathbb{R}^3)} + \left\| \left(\frac{r}{\log r} \right)^{1/2} f_\theta \right\|_{L^2(\mathbb{R}^3)} + \|r^2 \text{curl } \mathbf{f}\|_{L^2(\mathbb{R}^3)} \leq M \quad (1.11)$$

for some constant M . Then the axially symmetric solution (\mathbf{u}, p) to (1.1) with finite Dirichlet integral satisfied

$$|u_\theta(r, z)| \leq c(M) \left(\frac{\log r}{r} \right)^{\frac{1}{2}}. \quad (1.12)$$

for large r . Moreover, we have the following estimates for $\nabla \mathbf{u}$:

$$|\omega_\theta(r, z)| \leq c(M) r^{-(\frac{19}{16})^-}, \quad (1.13)$$

$$|\nabla u_r(r, z)| + |\nabla u_z(r, z)| \leq c(M) r^{-(\frac{9}{8})^-}, \quad (1.14)$$

$$|\omega_r(r, z)| + |\omega_z(r, z)| \leq c(M) r^{-(\frac{67}{64})^-}, \quad (1.15)$$

$$|\nabla u_\theta(r, z)| \leq c(M) r^{-(\frac{67}{64})^-}, \quad (1.16)$$

where we denote a^- to be any constant less than a .

Another main result in this paper is the first decay rates estimate in the Oz -direction for the D-solution of (1.1) without swirl. We realize that it is possible to derive the weighted energy estimates of $\Omega := \frac{\omega_\theta}{r}$ in the Oz direction. Together with previous weighted energy estimates on ω_θ , we can derive the decay rate of ω_θ with respect to $\rho = \sqrt{r^2 + z^2}$, which also yields the decay rate of \mathbf{u} , since \mathbf{u} has an integral representation formula in terms of ω_θ .

Theorem 1.2. Suppose $\mathbf{f} \in H^1(\Omega)$ be an axially symmetric vector field without swirl, and satisfying (1.11) and

$$\left\| (|z| + 1) \frac{(\partial_z f_r - \partial_r f_z)}{r} \right\|_{L^2(\mathbb{R}^3)} \leq M. \quad (1.17)$$

Then the axially symmetric D-solution (\mathbf{u}, p) to (1.1) without swirl satisfied

$$|\mathbf{u}(r, z)| \leq \frac{c(M)}{\rho^{(\frac{5}{48})^-}}, \quad (1.18)$$

$$|\omega(r, z)| \leq \frac{c(M)}{\rho^{(\frac{5}{16})^-}} \quad (1.19)$$

where $\rho = \sqrt{r^2 + z^2}$.

After this introduction section, some preliminary tools including a decay lemma and A_p weight for singular integral operators will be introduced. Then we prove Theorem 1.1 in Section 3. We first improve the weighted energy estimates and the decay rates of ω_θ by a bootstrap argument, these yields some good weighted estimates and decay rates of ∇u_r and ∇u_z by employing the A_p weight method and the decay lemma. Based on these and the equations of ω_r and ω_z , one can obtain some decay rates of ω_r and ω_z , which yields better decay rates of u_θ . In Section 4, we first realizes it is possible to obtain some weighted energy estimates of Ω in the Oz direction, which enables us to get some decay properties of \mathbf{u} in the Oz direction.

2 Preliminary

2.1 A decay lemma

The following decay lemma is proved by the techniques developed in [16], [17] and [7].

Lemma 2.1. *Suppose an smooth axially symmetric function $f(x)$ satisfies the following weighted energy estimates*

$$\int_{\mathbb{R}^3} r^{e_1} |f(r, z)|^2 dx \leq C, \quad (2.1)$$

$$\int_{\mathbb{R}^3} r^{e_2} |\nabla f(r, z)|^2 dx \leq C, \quad (2.2)$$

$$\int_{\mathbb{R}^3} r^{e_3} |\nabla \partial_z f(r, z)|^2 dx \leq C \quad (2.3)$$

with nonnegative constants e_1, e_2, e_3 . Then for any $r > 0$, we have

$$\int_{-\infty}^{\infty} |f(r, z)|^2 dz \leq C r^{-\frac{1}{2}(e_1+e_2)-1}, \quad (2.4)$$

$$\int_{-\infty}^{\infty} |\partial_z f(r, z)|^2 dz \leq C r^{-\frac{1}{2}(e_2+e_3)-1}, \quad (2.5)$$

$$|f(r, z)|^2 \leq C r^{-\frac{1}{4}(e_1+2e_2+e_3)-1}, \quad \forall z \in \mathbb{R}^3. \quad (2.6)$$

Proof. For any integer $n \geq 0$, from (2.1), we have

$$\int_{2^n}^{2^{n+1}} \int_{-\infty}^{\infty} r^{e_1} |f(r, z)|^2 r dr dz \leq C.$$

By the intermediate value theorem, there exists $r_n \in [2^n, 2^{n+1}]$ such that

$$\int_{-\infty}^{\infty} |f(r_n, z)|^2 dz \leq C r_n^{-e_1-2}.$$

For $\forall r > 0$, choose $r_n > r$ and then

$$\begin{aligned} \int_{-\infty}^{\infty} |f(r, z)|^2 dz &= \int_{-\infty}^{\infty} |f(r_n, z)|^2 dz - 2 \int_r^{r_n} \int_{-\infty}^{\infty} f(s, z) \partial_s f(s, z) ds dz := I_1 + I_2, \\ |I_2| &\leq C \left(\int_r^{r_n} \int_{-\infty}^{\infty} \frac{|f(s, z)|^2}{s^2} ds dz \right)^{\frac{1}{2}} \left(\int_r^{r_n} \int_{-\infty}^{\infty} |\partial_s f(s, z)|^2 ds dz \right)^{\frac{1}{2}} \\ &\leq \frac{C}{r^{\frac{1}{2}(e_1+e_2+2)}} \left(\int_r^{\infty} \int_{-\infty}^{\infty} s^{e_1} |f(s, z)|^2 ds dz \right)^{\frac{1}{2}} \left(\int_r^{\infty} \int_{-\infty}^{\infty} s^{e_2} |\partial_s f(s, z)|^2 ds dz \right)^{\frac{1}{2}} \\ &\leq O(r^{-\frac{1}{2}(e_1+e_2)-1}). \end{aligned}$$

Sending $n \rightarrow \infty$, $I_1 \rightarrow 0$. Hence we arrive at (2.4). Similarly, we also have (2.5).

To prove (2.6), for $|z| \leq z_1$ for some $z_1 \geq 1$, we use

$$|f(r, z)|^2 = \frac{1}{2z_1} \int_{-z_1}^{z_1} |f(r, t)|^2 dt + \left(|f(r, z)|^2 - \frac{1}{2z_1} \int_{-z_1}^{z_1} |f(r, t)|^2 dt \right) = J_1 + J_2.$$

By the mean value theorem and (2.4)-(2.5), it follows

$$\begin{aligned}
|J_2| &= (|f(r, z)|^2 - |f(r, z_*)|^2) \leq \int_{-z_1}^{z_1} \left| \frac{\partial}{\partial t} |f(r, t)|^2 \right| dt \\
&\leq c \left(\int_{-\infty}^{\infty} |f(r, t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\partial_t f(r, t)|^2 dt \right)^{\frac{1}{2}} \\
&\leq C r^{-\frac{1}{4}(e_1 + 2e_2 + e_3) - 1}, \\
J_1 &\leq \frac{C}{z_1 r^{\frac{1}{2}(e_1 + e_2) + 2}}.
\end{aligned}$$

Then letting z_1 goes to ∞ , we obtain (2.6). \square

2.2 A_p weight and singular integral operator

We first give the classical definition of A_p weight.

Definition 2.2. Let $p \in (1, \infty)$. A real valued function $w(x)$ is said to be in A_p class if it satisfies

$$\sup_{B \subset \mathbb{R}^3} \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty, \quad (2.7)$$

where the supremum is taken over all balls B in \mathbb{R}^3 . Here p' is the Hölder conjugate of p , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$

For function $w(x) \in A_p$, we can extend the Calderon-Zygmund inequality for the singular integral operator with the integral having weight function $w(x)$.

Theorem 2.3. ([24] page 205.) Let $p \in (1, \infty)$. Suppose T is a singular integral operator of the convolution type, and $w(x) \in A_p$. Then for $f \in L^p(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^3} |f(x)|^p w(x) dx. \quad (2.8)$$

Lemma 2.4. For any $p \in (1, \infty)$, the function $w(x) = r^\alpha$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\alpha \in (-2, 2(p-1))$, is in A_p class.

Proof. Similar results has been proved in [2]. Let $x_0 \in \mathbb{R}^3$ be given. Set $B = B_s(x_0)$ and $d = \sqrt{x_{01}^2 + x_{02}^2}$. If $d \geq 2s$, then $d - s \leq \sqrt{x_1^2 + x_2^2} \leq d + s$ for all $x \in B$. Thus if $\alpha \geq 0$

$$\begin{aligned}
L &:= \sup_{B \subset \mathbb{R}^3} \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} \\
&\leq \left(\frac{3}{4\pi r^3} \int_B (d+s)^\alpha dx \right) \left(\frac{3}{4\pi r^3} \int_B (d-s)^{-\frac{\alpha}{p-1}} dx \right)^{(p-1)} \\
&= C_2 \frac{(d+s)^\alpha}{(d-s)^\alpha} \leq C_3.
\end{aligned}$$

For $\alpha < 0$, one can also show that $L \leq C_3$. If $d < 2s$, then the cylinder $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 < (d+s)^2, |x_3 - x_3^0| < s\}$ contains the ball B . Thus if $\alpha \in (-2, 2(p-1))$,

$$\begin{aligned} L &\leq \left(\frac{3}{4\pi s^3} 2\pi \int_{x_3^0-s}^{x_3^0+s} \int_0^{d+s} \rho^\alpha \rho d\rho dx_3 \right) \times \left(\frac{3}{4\pi s^3} 2\pi \int_{x_3^0-s}^{x_3^0+s} \int_0^{d+s} \rho^{-\frac{\alpha}{p-1}+1} d\rho dx_3 \right)^{(p-1)} \\ &\leq \left(\frac{3}{2s^3} \frac{2}{2+\alpha} s(d+s)^{2+\alpha} \right) \left(\frac{3}{2s^3} \frac{2}{2-\frac{\alpha}{p-2}} s(d+s)^{2-\frac{\alpha}{p-1}} \right)^{(p-1)} \\ &\leq C(\alpha, p) < \infty. \end{aligned}$$

The proof is completed. \square

3 Proof of Theorem 1.1.

3.1 Improved estimates for ω_θ .

Since $\mathbf{f} \in H^{-1}(\mathbb{R}^3)$, the standard existence theory tells us that there exists a weak solution \mathbf{u} to (1.1) with finite Dirichlet integral. If $\mathbf{f} \in H^1(\mathbb{R}^3)$, by the L^q estimates of the Stokes system, then $\nabla^2 \mathbf{u} \in L^2(\mathbb{R}^3)$ and $\mathbf{u} \in L^\infty(\mathbb{R}^3)$. We first give the following basic decay estimates for D -solutions to (1.1).

Lemma 3.1. *Let (\mathbf{u}, p) be an axially symmetric smooth D -solutions to (1.1) with the forcing term \mathbf{f} satisfying (1.11). Then there is a constant $c(M)$ such that*

$$\int_{-\infty}^{\infty} |u_r(r, z)|^2 + |u_\theta(r, z)|^2 dz \leq c(M), \quad (3.1)$$

$$\int_{-\infty}^{\infty} |u_z(r, z)|^4 dz \leq \frac{c(M)}{r}, \quad (3.2)$$

$$\int_{-\infty}^{\infty} |\omega_\theta(r, z)|^2 dz \leq \frac{c(M)}{r}. \quad (3.3)$$

Proof. The inequality (3.1) was proven in Lemma 3.2 of [7]. In their proof, they had to used that $(\frac{u_r}{r}, \frac{u_\theta}{r}) \in L^2(\mathbb{R}^3)$. However, since $u_z(0, z)$ may not be zero, $\frac{u_z(r, z)}{r}$ may not belong to $L^2(\mathbb{R}^3)$. For (3.2), we see

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} |u_z(r, z)|^4 dz \right| = \left| -4 \int_{-\infty}^{\infty} \int_r^{\infty} |u_z(s, z)|^2 u_z(s, z) \partial_s u_z(s, z) ds dz \right| \\ &\leq \frac{4}{r} \int_{-\infty}^{\infty} \int_r^{\infty} |u_z(s, z)|^3 |\partial_s u_z(s, z)| s ds dz \\ &\leq \frac{c}{r} \left(\int_{-\infty}^{\infty} \int_0^{\infty} |u_z(s, z)|^6 s ds dz \right)^{1/2} \left(\int_{-\infty}^{\infty} \int_0^{\infty} |\partial_s u_z(s, z)|^2 s ds dz \right)^{1/2} \\ &\leq \frac{c(M)}{r}. \end{aligned}$$

The last estimate (3.3) follows from (2.4) in Lemma 2.1, since ω_θ and $\nabla \omega_\theta$ belong to $L^2(\mathbb{R}^3)$. \square

Lemma 3.2. *Let (\mathbf{u}, p) be an axially symmetric smooth D -solutions to (1.1) with the forcing term \mathbf{f} satisfying (1.11). Suppose that*

$$|u_r(r, z)| + |u_\theta(r, z)| + |u_z(r, z)| \leq Cr^{-\delta}, \quad \forall z \in \mathbb{R} \quad (3.4)$$

holds for some $\delta \in [0, 1]$. Then the following estimates holds

$$\int_{\mathbb{R}^3} |\omega_\theta|^2 dx \leq c(M), \quad (3.5)$$

$$\int_{\mathbb{R}^3} r^{1+\delta} |\nabla \omega_\theta|^2 dx \leq c(M), \quad (3.6)$$

$$\int_{\mathbb{R}^3} r^{1+3\delta} |\partial_z \nabla \omega_\theta|^2 dx \leq c(M). \quad (3.7)$$

In particular, by Lemma 2.1, we obtain the following decay rate for ω_θ :

$$|\omega_\theta(r, z)| \leq C(M) r^{-\frac{7}{8} - \frac{5}{8}\delta}. \quad (3.8)$$

Proof. Since $\nabla \mathbf{u} \in L^2(\mathbb{R}^3)$, then (3.5) holds. Take a cut-off function $\eta \in C_0^\infty(\mathbb{R}^3)$, $\eta = \eta(\rho)$, $\rho = \sqrt{r^2 + z^2}$, satisfying $0 \leq \eta \leq 1$, $\eta(\rho) = 1$ on $0 \leq \rho \leq \rho_0$, $\eta(\rho) = 0$ for $\rho \geq 2\rho_0$, so that $|\nabla \eta| \leq \frac{\epsilon}{\rho}$ for $\rho_0 \leq \rho \leq 2\rho_0$. Then $\rho|\eta'(\rho)| \leq c$ for $\forall \rho \geq 0$. Multiplying (1.5) by $\eta^2 r^{a_1} \omega_\theta$ and integrating over \mathbb{R}^3 , we get an integral identity with left and right hand sides as

$$\begin{aligned} LHS &= -\pi \int_{-\infty}^{\infty} \int_0^{\infty} r^{a_1} 2\eta\eta' \frac{ru_r + zu_z}{\sqrt{r^2 + z^2}} \omega_\theta^2 r dr dz - \pi \int_{-\infty}^{\infty} \int_0^{\infty} a_1 r^{a_1-1} \eta^2 \omega_\theta^2 u_r r dr dz \\ &\quad - \int_{\mathbb{R}^3} r^{a_1-1} \eta^2 u_r \omega_\theta^2 dx - 2 \int_{\mathbb{R}^3} r^{a_1-2} \eta^2 u_\theta \partial_z u_\theta \omega_\theta dx := \sum_{k=1}^4 I_k, \\ RHS &= -2\pi \int_{-\infty}^{\infty} \int_0^{\infty} r^{a_1} |\nabla \omega_\theta|^2 \eta^2 r dr dz - 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} 2r^{a_1} \omega_\theta \eta\eta' \frac{r\partial_r \omega_\theta + z\partial_z \omega_\theta}{\sqrt{r^2 + z^2}} r dr dz \\ &\quad - a_1 \int_{\mathbb{R}^3} r^{a_1-1} \eta^2 \omega_\theta \partial_r \omega_\theta r dr dz - \int_{\mathbb{R}^3} r^{a_1-2} \eta^2 \omega_\theta^2 dx + \int_{\mathbb{R}^3} r^{a_1} \eta^2 \omega_\theta (\partial_z f_r - \partial_r f_z) dx \\ &:= \sum_{k=1}^5 J_k. \end{aligned}$$

Then for $a_1 = 1 + \delta$, we have the following estimates

$$\begin{aligned} \sum_{j=1}^3 |I_j| &\leq C \int_{\mathbb{R}^3} (|u_r| + |u_z|) r^{a_1-1} \omega_\theta^2 dx \leq C \|(u_r, u_z) r^{a_1-1}\|_{L^\infty} \|\omega_\theta\|_{L^2}^2, \\ |I_4| &\leq C \int_{\mathbb{R}^3} \eta r^{a_1-1} |u_\theta| |\partial_z u_\theta| |\omega_\theta| dx \leq C \|r^{a_1-1} u_\theta\|_{L^\infty} \|\partial_z u_\theta\|_{L^2} \|\omega_\theta\|_{L^2}, \\ \sum_{k=2}^3 |J_k| &\leq \int_{\mathbb{R}^3} |r\eta'| \left| \frac{\omega_\theta}{r} \right|^{1-\frac{a_1}{2}} |\omega_\theta|^{\frac{a_1}{2}} (r^{\frac{a_1}{2}} \eta |\nabla \omega_\theta|) dx \\ &\leq \epsilon \|\eta r^{\frac{a_1}{2}} \nabla \omega_\theta\|_{L^2}^2 + C(\epsilon) \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^{2-a_1} \|\omega_\theta\|_{L^2}^{a_1}, \\ |J_4| &\leq \int_{\mathbb{R}^3} \left| \frac{\omega_\theta}{r} \right|^{2-a_1} |\omega_\theta|^{a_1} dx \leq \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^{2-a_1} \|\omega_\theta\|_{L^2}^{a_1}, \quad \text{if } 0 \leq a_1 \leq 2, \\ |J_5| &\leq C \|\omega_\theta\|_{L^2(\mathbb{R}^3)} \|r^2 (\partial_z f_r - \partial_r f_z)\|_{L^2}, \quad \text{if } 0 \leq a_1 \leq 2. \end{aligned}$$

Assuming (3.4), letting $\rho_0 \rightarrow \infty$, then one obtains (3.5) and (3.6).

Furthermore, we can obtain (3.7) by using the equation for $\partial_z \omega_\theta$:

$$\partial_z \left((u_r \partial_r + u_z \partial_z) \omega_\theta - \frac{u_r}{r} \omega_\theta - \frac{1}{r} \partial_z (u_\theta^2) \right) = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) \partial_z \omega_\theta + \partial_z (\partial_z f_r - \partial_r f_z). \quad (3.9)$$

Multiplying (3.9) by $\eta^2 r^{b_1} \partial_z \omega_\theta$ and integrating over \mathbb{R}^3 , we get an integral identity with left and right hand sides as

$$\begin{aligned} LHS &= - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \omega_\theta \eta^2 r^{b_1} \partial_z^2 \omega_\theta dx - 2 \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \omega_\theta r^{b_1} \eta \eta' \frac{z \partial_z \omega_\theta}{\sqrt{r^2 + z^2}} dx \\ &\quad + \int_{\mathbb{R}^3} r^{b_1-1} u_r \eta^2 \omega_\theta \partial_z^2 \omega_\theta dx + 2 \int_{\mathbb{R}^3} r^{b_1-1} u_r \omega_\theta \eta \eta' \frac{z}{\sqrt{r^2 + z^2}} \partial_z \omega_\theta dx \\ &\quad + \int_{\mathbb{R}^3} 2r^{b_1-2} u_\theta \partial_z u_\theta \eta^2 \partial_z^2 \omega_\theta dx + 4 \int_{\mathbb{R}^3} r^{b_1-1} u_\theta \partial_z u_\theta \eta \eta' \frac{z \partial_z \omega_\theta}{\sqrt{r^2 + z^2}} dx := \sum_{k=1}^6 I'_k, \\ RHS &= - \int_{\mathbb{R}^3} r^{b_1} \eta^2 |\nabla \partial_z \omega_\theta|^2 dx - \int_{\mathbb{R}^3} 2r^{b_1} \eta \eta' \partial_z \omega_\theta \frac{r \partial_r^2 \omega_\theta + z \partial_z^2 \omega_\theta}{\sqrt{r^2 + z^2}} dx - b_1 \int_{\mathbb{R}^3} r^{b_1-1} \eta^2 \partial_z \omega_\theta \partial_{r^2}^2 \omega_\theta dx \\ &\quad - \int_{\mathbb{R}^3} r^{b_1-2} \eta^2 (\partial_z \omega_\theta)^2 dx - \int_{\mathbb{R}^3} r^{b_1} \partial_z (\eta^2 \partial_z \omega_\theta) (\partial_z f_r - \partial_r f_z) dx := \sum_{k=1}^5 J'_k. \end{aligned}$$

We estimate these terms as follows.

$$\begin{aligned} \sum_{k=2}^3 |J'_k| &\leq \|\eta r^{b_1/2} |\nabla \partial_z \omega_\theta|\|_{L^2} \|r^{b_1/2-1} \partial_z \omega_\theta\|_{L^2} \leq \epsilon \|\eta r^{b_1/2} |\nabla \partial_z \omega_\theta|\|_{L^2}^2 + C \|r^{a_1/2} \partial_z \omega_\theta\|_{L^2}^2, \text{ if } b_1/2 - 1 \leq a_1/2, \\ |J'_4| &\leq \|r^{b_1/2-1} |\nabla \omega_\theta|\|_{L^2}^2 \leq C \|r^{a_1/2} |\nabla \omega_\theta|\|_{L^2}^2, \text{ if } b_1/2 - 1 \leq a_1/2, \\ |J'_5| &\leq C \|\eta r^{b_1/2} \partial_z^2 \omega_\theta\|_{L^2} \|r^{b_1/2} (\partial_z f_r - \partial_r f_z)\|_{L^2} + \|r^{a_1/2} \partial_z \omega_\theta\|_{L^2} \|r^{b_1-a_1/2-1} (\partial_z f_r - \partial_r f_z)\|_{L^2} \\ &\leq \epsilon \|\eta r^{b_1/2} \partial_z^2 \omega_\theta\|_{L^2}^2 + C(\epsilon) \|r^2 (\partial_z f_r - \partial_r f_z)\|_{L^2}^2 + \|r^{a_1/2} \partial_z \omega_\theta\|_{L^2} \|r^2 (\partial_z f_r - \partial_r f_z)\|_{L^2}, \text{ if } b \leq 4, \\ |I'_1| &\leq \int_{\mathbb{R}^3} \eta r^{b_1/2-a_1/2} (|u_r| + |u_z|) r^{a_1/2} |\nabla \omega_\theta| \cdot (\eta r^{b_1/2} |\partial_z^2 \omega_\theta|) dx \\ &\leq \epsilon \|r^{b_1/2} \eta \nabla \partial_z \omega_\theta\|_{L^2}^2 + C(\epsilon) \|r^{a_1/2} \nabla \omega_\theta\|_{L^2}^2, \text{ if } b_1/2 - a_1/2 \leq \delta, \\ |I'_2| &\leq 2 \int_{\mathbb{R}^3} |r \eta'| \eta r^{b_1-a_1-1} (|u_r| + |u_z|) r^{a_1} |\nabla \omega_\theta|^2 dx \leq C \|r^{a_1/2} |\nabla \omega_\theta|\|_{L^2}^2, \text{ if } b_1 - a_1 - 1 \leq \delta, \\ |I'_3| &\leq \int_{\mathbb{R}^3} r^{b_1/2-1} |u_r| \omega_\theta (\eta r^{b_1/2} |\partial_z^2 \omega_\theta|) dx \leq \epsilon \|\eta r^{b_1/2} |\nabla \partial_z \omega_\theta|\|_{L^2}^2 + C \|\omega_\theta\|_{L^2}^2, \text{ if } b_1/2 - 1 \leq \delta, \\ |I'_4| &\leq \int_{\mathbb{R}^3} r^{b_1-a_1/2-2} |u_r| \omega_\theta (\eta r^{a_1/2} |\partial_z \omega_\theta|) dx \leq C \|\omega_\theta\|_{L^2} \|r^{a_1/2} \eta |\nabla \omega_\theta|\|_{L^2}, \text{ if } b_1 - a_1/2 - 2 \leq \delta, \\ |I'_5| &\leq 2 \int_{\mathbb{R}^3} \eta r^{b_1/2-1} |u_\theta| |\partial_z u_\theta| \eta r^{b_1/2} \partial_z^2 \omega_\theta dx \leq \epsilon \|r^{b_1/2} \eta \nabla \partial_z \omega_\theta\|_{L^2}^2 + C(\epsilon) \|\nabla u_\theta\|_{L^2}^2, \text{ if } b_1/2 - 1 \leq \delta \\ |I'_6| &\leq 2 \int_{\mathbb{R}^3} |r \eta'| r^{b_1-a_1/2-2} |u_\theta| |\partial_z u_\theta| \eta r^{a_1/2} \partial_z \omega_\theta dx \leq C \|\partial_z u_\theta\|_{L^2} \|r^{a_1/2} \partial_z \omega_\theta\|_{L^2}, \text{ if } b_1 - a_1/2 - 2 \leq \delta. \end{aligned}$$

Then it is easy to see the essential restriction on b_2 is $b_2/2 - a_2/2 \leq \delta$, i.e. $b_2 \leq 1 + 3\delta$. Taking $b_2 = 1 + 3\delta$ and letting $\rho_0 \rightarrow \infty$, then we obtain (3.7).

□

Remark 3.3. By (1.8)-(1.9), we see that (3.4) holds for any $0 \leq \delta < \frac{3}{8}$. Hence (3.8) implies that

$$|\omega_\theta(r, z)| \leq C(M)r^{-(\frac{71}{64})^-}. \quad (3.10)$$

Remark 3.4. The case $\delta = 0$ was obtained in [7]. We can also extend these arguments to the exterior domain case by choosing the cut-off function η as ϕ in [7]. Suppose $0 \in \Omega^c$, then choose $\rho_1 > \text{diam } \Omega$, define $\eta = \eta(\rho) \in C_0^\infty(\mathbb{R}^3)$, $\rho = \sqrt{r^2 + z^2}$, satisfying $0 \leq \eta \leq 1$, $\eta(\rho) = 1$ on $2\rho_1 \leq \rho \leq 2\rho_2$, $\eta(\rho) = 0$ for $\rho \geq 2\rho_2$ or $0 \leq \rho \leq \rho_1$, so that $|\nabla \eta| \leq \frac{c}{\rho}$ for $\rho_1 \leq \rho \leq 2\rho_1$ and $\rho_2 \leq \rho \leq 2\rho_2$.

Lemma 3.5. Let (\mathbf{u}, p) be an axially symmetric smooth D-solutions to (1.1) with the forcing term \mathbf{f} satisfying (1.11). Suppose that

$$|u_r(r, z)| + |u_\theta(r, z)| + |u_z(r, z)| \leq Cr^{-\delta}, \quad \forall z \in \mathbb{R} \quad (3.11)$$

holds for some $\delta \in [0, 1]$. Then the following estimates holds

$$\int_{\mathbb{R}^3} (|\nabla u_r|^2 + |\nabla u_z|^2) dx \leq c(M), \quad (3.12)$$

$$\int_{\mathbb{R}^3} r^{1+\delta_1} (|\nabla^2 u_r|^2 + |\nabla^2 u_z|^2) dx \leq c(M), \quad (3.13)$$

$$\int_{\mathbb{R}^3} r^{1+\delta_2} (|\partial_z \nabla^2 u_r|^2 + |\partial_z \nabla^2 u_z|^2) dx \leq c(M), \quad (3.14)$$

where $\delta_1 = \delta$ if $\delta \in [0, 1)$ and $\delta_1 = 1^-$ if $\delta = 1$, $\delta_2 = 3\delta$ if $\delta < \frac{1}{3}$ and $\delta_2 = 1^-$ if $\delta \in [\frac{1}{3}, 1]$.

In particular, by Lemma 2.1, we obtain the following decay rate for ω_θ :

$$|\nabla u_r(r, z)| + |\nabla u_z(r, z)| \leq C(M)r^{-\frac{7}{8} - \frac{2\delta_1 + \delta_2}{8}}. \quad (3.15)$$

Proof. Since $-\Delta(u_r \mathbf{e}_r + u_z \mathbf{e}_z) = \text{curl}(\omega_\theta \mathbf{e}_\theta)$, then

$$\nabla(u_r \mathbf{e}_r + u_z \mathbf{e}_z) = \nabla(-\Delta)^{-1} \text{curl}(\omega_\theta \mathbf{e}_\theta).$$

Hence ∇u_r and ∇u_z can be expressed as singular integral operators of ω_θ , we can apply Lemma 2.4 to complete the proof. \square

Remark 3.6. By (1.8)-(1.9), we take $\delta = (\frac{3}{8})^-$, then we have

$$|\nabla u_r(r, z)| + |\nabla u_z(r, z)| \leq C(M)r^{-(\frac{35}{32})^-}. \quad (3.16)$$

Remark 3.7. It seems difficult to derive weighted estimates of (u_r, u_z) by using the equations of u_r and u_z directly, since we do not have good estimates on the pressure.

3.2 Estimates for ω_r and ω_z

Lemma 3.8. Let (\mathbf{u}, p) be an axially symmetric smooth D-solutions to (1.1) with the forcing term \mathbf{f} satisfying (1.11). Suppose that

$$|u_r(r, z)| + |u_z(r, z)| \leq Cr^{-\delta}, \quad (3.17)$$

$$|\nabla u_r(r, z)| + |\nabla u_z(r, z)| \leq Cr^{-1-\gamma} \quad (3.18)$$

holds for some $\delta, \gamma \in [0, 1]$. Then the following estimates holds

$$\int_{\mathbb{R}^3} (|\omega_r|^2 + |\omega_z|^2) dx \leq c(M), \quad (3.19)$$

$$\int_{\mathbb{R}^3} r^{1+\delta \wedge \gamma} (|\nabla \omega_r|^2 + |\nabla \omega_z|^2) dx \leq c(M), \quad (3.20)$$

$$\int_{\mathbb{R}^3} r^{1+\delta \wedge \gamma + 2\delta} (|\partial_z \nabla \omega_r|^2 + |\partial_z \nabla \omega_z|^2) dx \leq c(M), \quad (3.21)$$

where $\delta \wedge \gamma = \min\{\delta, \gamma\}$. In particular, by Lemma 2.1, we obtain the following decay rate for ω_θ :

$$|\omega_r(r, z)| + |\omega_z(r, z)| \leq C(M) r^{-\frac{7}{8} - \frac{1}{8}(3(\delta \wedge \gamma) + 2\delta)}. \quad (3.22)$$

Remark 3.9. By (1.8)-(1.9) and (3.16), we take $\delta = (\frac{3}{8})^-$ and $\gamma = (\frac{3}{32})^-$ in Lemma 3.8, then

$$|\omega_r(r, z)| + |\omega_z(r, z)| \leq C(M) r^{-(\frac{257}{256})^-}. \quad (3.23)$$

Proof. Multiplying (1.4) and (1.6) by $\eta^2 r^{a_2} \omega_r$ and $\eta^2 r^{a_2} \omega_z$ respectively, integrating over \mathbb{R}^3 and adding them together, we get an integral identity with left and right hand sides as

$$\begin{aligned} LHS &= -\frac{1}{2} \int_{\mathbb{R}^3} a_2 r^{a_2-1} \eta^2 u_r (|\omega_r|^2 + |\omega_z|^2) dx - \int_{\mathbb{R}^3} r^{a_2} \eta \eta' (|\omega_r|^2 + |\omega_z|^2) \frac{r u_r + z u_z}{\sqrt{r^2 + z^2}} dx \\ &\quad - \int_{\mathbb{R}^3} \eta^2 r^{a_2} [\omega_r (\omega_r \partial_r + \omega_z \partial_z) u_r + \omega_z (\omega_r \partial_r + \omega_z \partial_z) u_z] dx := A_1 + A_2 + A_3, \\ RHS &= - \int_{\mathbb{R}^3} \eta^2 r^{a_2} (|\nabla \omega_r|^2 + |\nabla \omega_z|^2) dx - 2 \int_{\mathbb{R}^3} \eta \eta' r^{a_2} \omega_r \frac{r \partial_r \omega_r + z \partial_z \omega_r}{\sqrt{r^2 + z^2}} dx \\ &\quad - 2 \int_{\mathbb{R}^3} \eta \eta' r^{a_2} \omega_z \frac{r \partial_r \omega_z + z \partial_z \omega_z}{\sqrt{r^2 + z^2}} dx - a_2 \int_{\mathbb{R}^3} \eta^2 r^{a_2-1} \omega_r \partial_r \omega_r dx - a_2 \int_{\mathbb{R}^3} r^{a_2-1} \eta^2 \omega_z \partial_r \omega_z dx \\ &\quad - \int_{\mathbb{R}^3} \eta^2 r^{a_2-2} \omega_r^2 dx + \int_{\mathbb{R}^3} \eta^2 r^{a_2} [-\omega_r \partial_z f_\theta + \frac{\omega_z}{r} \partial_r (r f_\theta)] dx := \sum_{k=1}^7 B_k. \end{aligned}$$

Take $a_2 = 1 + \delta \wedge \gamma$, we can estimate $I_i, i = 1, 2, 3$ and $J_k, k = 1, \dots, 6$ as follows.

$$\begin{aligned} \sum_{k=1}^2 |A_k| &\leq C(\|r^{a_2-1} u_r\|_{L^\infty} + \|r^{a_2-1} u_z\|_{L^\infty})(\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2) \leq C(\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2), \\ |A_3| &\leq C(\|r^{a_2} \nabla u_r\|_{L^\infty} + \|r^{a_2} \nabla u_z\|_{L^\infty})(\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2) \leq C(\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2), \\ \sum_{k=2}^5 |B_k| &\leq C \int_{\mathbb{R}^3} r^{a_2-1} (|\omega_r| + |\omega_z|) \eta (|\nabla \omega_r| + |\nabla \omega_z|) dx \\ &\leq \epsilon \int_{\mathbb{R}^3} \eta^2 r^{2a_2-2} (|\nabla \omega_r|^2 + |\nabla \omega_z|^2) dx + C(\epsilon)(\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2), \\ &\leq \epsilon \int_{\mathbb{R}^3} \eta^2 r^{a_2} (|\nabla \omega_r|^2 + |\nabla \omega_z|^2) dx + C(\epsilon)(\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2), \\ B_6 &\leq \|\omega_r\|_{L^2}^{\delta \wedge \gamma} \left\| \frac{\omega_r}{r} \right\|_{L^2}^{1-\delta \wedge \gamma}, \\ B_7 &\leq \|\omega_r\|_{L^2} \|r^2 \partial_z f_\theta\|_{L^2} + \|\omega_z\|_{L^2} \|r^2 (\partial_r f_\theta + \frac{f_\theta}{r})\|_{L^2}. \end{aligned}$$

Letting $\rho_0 \rightarrow \infty$, we have proved (3.20).

To show (3.21), we need to use the equations for $\partial_z \omega_r$ and $\partial_z \omega_z$:

$$\partial_z \left((u_r \partial_r + u_z \partial_z) \omega_r - (\omega_r \partial_r + \omega_z \partial_z) u_r \right) = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) \partial_z \omega_r - \partial_z^2 f_\theta, \quad (3.24)$$

$$\partial \left((u_r \partial_r + u_z \partial_z) \omega_z - (\omega_r \partial_r + \omega_z \partial_z) u_z \right) = (\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2) \partial_z \omega_z + \frac{1}{r} \partial_{rz}^2 (r f_\theta). \quad (3.25)$$

Multiplying (3.24) and (3.25) by $\eta^2 r^{b_2} \partial_z \omega_r$ and $\eta^2 r^{b_2} \partial_z \omega_z$ respectively, integrating over \mathbb{R}^3 and adding them together, we get an integral identity with left and right hand sides as

$$\begin{aligned} LHS &= - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \omega_r \eta^2 r^{b_2} \partial_z^2 \omega_r dx - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \omega_z \eta^2 r^{b_2} \partial_z^2 \omega_z dx \\ &\quad - 2 \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \omega_r \eta' r^{b_2} \frac{z \partial_z \omega_r}{\sqrt{r^2 + z^2}} dx - 2 \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \omega_z \eta' r^{b_2} \frac{z \partial_z \omega_z}{\sqrt{r^2 + z^2}} dx \\ &\quad + \int_{\mathbb{R}^3} (\omega_r \partial_r + \omega_z \partial_z) u_r \eta^2 r^{b_2} \partial_z^2 \omega_r dx + \int_{\mathbb{R}^3} (\omega_r \partial_r + \omega_z \partial_z) u_z \eta^2 r^{b_2} \partial_z^2 \omega_z dx \\ &\quad + 2 \int_{\mathbb{R}^3} (\omega_r \partial_r + \omega_z \partial_z) u_r \eta' r^{b_2} \frac{z \partial_z \omega_r}{\sqrt{r^2 + z^2}} dx + 2 \int_{\mathbb{R}^3} (\omega_r \partial_r + \omega_z \partial_z) u_z \eta' r^{b_2} \frac{z \partial_z \omega_z}{\sqrt{r^2 + z^2}} dx \\ &:= \sum_{i=1}^8 A'_i, \\ RHS &= - \int_{\mathbb{R}^3} \eta^2 r^{b_2} (|\nabla \partial_z \omega_r|^2 + |\nabla \partial_z \omega_z|^2) dx - 2 \int_{\mathbb{R}^3} \eta' r^{b_2} \partial_z \omega_r \frac{r \partial_{rz}^2 \omega_r + z \partial_z^2 \omega_r}{\sqrt{r^2 + z^2}} dx \\ &\quad - 2 \int_{\mathbb{R}^3} \eta' r^{b_2} \partial_z \omega_z \frac{r \partial_{rz}^2 \omega_z + z \partial_z^2 \omega_z}{\sqrt{r^2 + z^2}} dx - b_2 \int_{\mathbb{R}^3} \eta^2 r^{b_2-1} \partial_z \omega_r \partial_{rz}^2 \omega_r dx \\ &\quad - b_2 \int_{\mathbb{R}^3} r^{b_2-1} \eta^2 \partial_z \omega_z \partial_{rz}^2 \omega_z dx - \int_{\mathbb{R}^3} \eta^2 r^{b_2-2} (\partial_z \omega_r)^2 dx \\ &\quad - \int_{\mathbb{R}^3} r^{b_2} [-\partial_z (\eta^2 \partial_z \omega_r) \partial_z f_\theta + \partial_z (\eta^2 \partial_z \omega_z) \frac{1}{r} \partial_r (r f_\theta)] dx := \sum_{k=1}^7 B'_k. \end{aligned}$$

We estimate these terms as follows.

$$\begin{aligned} \sum_{k=2}^5 |B'_k| &\leq \int_{\mathbb{R}^3} r^{\frac{b_2}{2}-1} (|\nabla \omega_r| + |\nabla \omega_z|) \eta r^{\frac{b_2}{2}} (|\nabla \partial_z \omega_r| + |\nabla \partial_z \omega_z|) dx \\ &\leq \epsilon \|\eta r^{\frac{b_2}{2}} (|\nabla \partial_z \omega_r| + |\nabla \partial_z \omega_z|)\|_{L^2}^2 + C(\epsilon) \|r^{\frac{a_2}{2}} (|\nabla \omega_r| + |\nabla \omega_z|)\|_{L^2}^2, \quad \text{if } \frac{b_2}{2} - 1 \leq a_2, \\ |B'_6| &\leq \|\eta r^{\frac{a_2}{2}} \nabla \omega_r\|_{L^2}^2, \quad \text{if } b_2 - 2 \leq a_2, \\ |B'_7| &\leq \|\eta r^{b_2/2} \partial_z^2 \omega_r\|_{L^2} \|r^{b_2/2} \partial_z f_\theta\|_{L^2} + \|\eta r^{b_2/2} \partial_z^2 \omega_z\|_{L^2} \|r^{b_2/2} (\partial_r f_\theta + \frac{1}{r} f_\theta)\|_{L^2} \\ &\quad + \|r^{a_2/2} \partial_z \omega_r\|_{L^2} \|r^{b_2-a_2/2-1} \partial_z f_\theta\|_{L^2} + \|r^{a_2/2} \partial_z \omega_z\|_{L^2} \|r^{b_2-a_2/2-1} (\partial_r f_\theta + \frac{1}{r} f_\theta)\|_{L^2} \\ &\leq \epsilon \|\eta r^{b_2/2} (\partial_z \nabla \omega_r, \partial_z \nabla \omega_z)\|_{L^2}^2 + \|r^{a_2/2} (\nabla \omega_r, \nabla \omega_z)\|_{L^2}^2 + C(\epsilon) \|r^2 (\partial_z f_\theta, \partial_r f_\theta + \frac{1}{r} f_\theta)\|_{L^2}^2. \end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^2 |A'_k| &\leq \int_{\mathbb{R}^3} r^{\frac{b_2}{2} - \frac{a_2}{2}} (|u_r| + |u_z|) \eta r^{\frac{a_2}{2}} (|\nabla \omega_r| + |\nabla \omega_z|) \eta r^{\frac{b_2}{2}} (|\nabla \partial_z \omega_r| + |\nabla \partial_z \omega_z|) dx \\
&\leq \epsilon \|\eta r^{\frac{b_2}{2}} (|\nabla \partial_z \omega_r| + |\nabla \partial_z \omega_z|)\|_{L^2}^2 + C \|\eta r^{\frac{a_2}{2}} (|\nabla \omega_r| + |\nabla \omega_z|)\|_{L^2}^2, \quad \text{if } \frac{b_2}{2} - \frac{a_2}{2} \leq \delta, \\
\sum_{k=3}^4 |A'_k| &\leq \int_{\mathbb{R}^3} r^{b_2 - a_2 - 1} (|u_r| + |u_z|) r^{a_2} (|\nabla \omega_r|^2 + |\nabla \omega_z|^2) dx \\
&\leq C \|r^{\frac{a_2}{2}} (|\nabla \omega_r| + |\nabla \omega_z|)\|_{L^2}^2, \quad \text{if } b_2 - a_2 - 1 \leq \delta, \\
\sum_{k=5}^6 |A'_k| &\leq \int_{\mathbb{R}^3} r^{\frac{b_2}{2}} (|\nabla u_r| + |\nabla u_z|) (|\omega_r| + |\omega_z|) \eta r^{\frac{b_2}{2}} (|\nabla \partial_z u_r| + |\nabla \partial_z u_z|) dx \\
&\leq \epsilon \|\eta r^{\frac{b_2}{2}} (|\nabla \partial_z u_r| + |\nabla \partial_z u_z|)\|_{L^2}^2 + C(\epsilon) (\|\omega_r\|_{L^2}^2 + \|\omega_z\|_{L^2}^2), \quad \text{if } \frac{b_2}{2} \leq 1 + \gamma, \\
\sum_{k=7}^8 |A'_k| &\leq \int_{\mathbb{R}^3} r^{b_2 - \frac{a_2}{2} - 1} (|\nabla u_r| + |\nabla u_z|) (|\omega_r| + |\omega_z|) \eta r^{\frac{a_2}{2}} (|\partial_z \omega_r| + |\partial_z \omega_z|) dx \\
&\leq C (\|\omega_r\|_{L^2} + \|\omega_z\|_{L^2}) \|\eta r^{\frac{a_2}{2}} (|\partial_z \omega_r| + |\partial_z \omega_z|)\|_{L^2}, \quad \text{if } b_2 - \frac{a_2}{2} - 1 \leq 1 + \gamma.
\end{aligned}$$

It is easy to see the essential restriction on b_2 is $\frac{b_2}{2} - \frac{a_2}{2} \leq \delta$, i.e. $b_2 \leq a_2 + 2\delta$. Hence we choose $b_2 = 1 + \delta \wedge \gamma + 2\delta$ and get (3.21) by letting $\rho_0 \rightarrow \infty$.

□

3.3 Improved decay estimates on u_θ

Since $\text{curl}(u_\theta \mathbf{e}_\theta) = \omega_r \mathbf{e}_r + \omega_z \mathbf{e}_z$ and $\text{div}(u_\theta \mathbf{e}_\theta) = 0$, then

$$-\Delta(u_\theta \mathbf{e}_\theta) = \text{curl}(\omega_r \mathbf{e}_r + \omega_z \mathbf{e}_z).$$

Fix $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$, we choose a smooth cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$, which is axially symmetric and satisfies $0 \leq \psi \leq 1$,

$$\psi(\rho, \kappa) = \begin{cases} 1, & r/2 < \rho < 2r, \quad -z_1 < \kappa < z_1, \\ 0, & \rho < r/4, \rho > 4r, \text{ or } |\kappa| > z_1, \end{cases}$$

where z_1 is any constant such that $z_1 > \max(2|z|, 1)$. Since

$$-\Delta(\psi u_\theta \mathbf{e}_\theta) = \psi \text{curl}(\omega_r \mathbf{e}_r + \omega_z \mathbf{e}_z) - 2\nabla \psi \cdot \nabla(u_\theta \mathbf{e}_\theta) - (\Delta \psi) u_\theta \mathbf{e}_\theta,$$

we get the integral representation for u_θ in terms of (ω_r, ω_z) and the fundamental solution $\Gamma(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$ of the Laplace operator: for $\mathbf{x} = (r \cos \theta, r \sin \theta, z)$

$$\begin{aligned}
(\psi u_\theta \mathbf{e}_\theta)(x) &= \int_{\mathbb{R}^3} \Gamma(x-y) \psi(y) \text{curl}(\omega_\rho \mathbf{e}_\rho + \omega_\kappa \mathbf{e}_\kappa) dy - 2 \int_{\mathbb{R}^3} \Gamma(x-y) \nabla \psi \cdot \nabla(u_\theta \mathbf{e}_\theta)(y) dy \\
&\quad - \int_{\mathbb{R}^3} \Gamma(x-y) (\Delta \psi)(y) (u_\theta \mathbf{e}_\theta)(y) dy \\
&= - \int_{\mathbb{R}^3} \nabla_y \Gamma(x-y) \times [\psi(y) (\omega_\rho \mathbf{e}_\rho + \omega_\kappa \mathbf{e}_\kappa)(y)] dy \\
&\quad - \int_{\mathbb{R}^3} \Gamma(x-y) (\nabla_y \psi)(y) \times (\omega_\rho \mathbf{e}_\rho + \omega_\kappa \mathbf{e}_\kappa)(y) dy \\
&\quad + 2 \int_{\mathbb{R}^3} [\nabla_y \Gamma(x-y) \cdot \nabla_y \psi(y)] (u_\theta \mathbf{e}_\theta)(y) dy + \int_{\mathbb{R}^3} \Gamma(x-y) \Delta_y \psi(y) (u_\theta \mathbf{e}_\theta)(y) dy.
\end{aligned} \tag{3.26}$$

By taking inner product to (3.26) by \mathbf{e}_θ , we get the following integral representation for u_θ : for $\mathbf{x} = (r \cos \theta, r \sin \theta, z)$

$$\begin{aligned}
u_\theta(r, z) &= - \int_{\mathbb{R}^3} \frac{\partial \hat{\Gamma}}{\partial \kappa} \psi(y) \omega_\rho(y) \cos(\theta - \phi) dy + \int_{\mathbb{R}^3} \frac{\partial \hat{\Gamma}}{\partial \rho} \psi(y) \omega_\kappa(y) \cos(\theta - \phi) dy \\
&\quad + \int_{\mathbb{R}^3} \frac{1}{\rho} \frac{\partial \hat{\Gamma}}{\partial \phi} \phi \omega_\kappa \sin(\theta - \phi) dy - \int_{\mathbb{R}^3} \hat{\Gamma} \frac{\partial \psi}{\partial \kappa} \omega_\rho \cos(\theta - \phi) dy \\
&\quad + \int_{\mathbb{R}^3} \hat{\Gamma} \frac{\partial \psi}{\partial \rho} \omega_\kappa \cos(\theta - \phi) dy + \int_{\mathbb{R}^3} \hat{\Gamma} \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \omega_\kappa \sin(\theta - \phi) dy \\
&\quad + 2 \int_{\mathbb{R}^3} \left(\frac{\partial \hat{\Gamma}}{\partial \rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial \hat{\Gamma}}{\partial \kappa} \frac{\partial \psi}{\partial \kappa} \right) u_\phi \cos(\theta - \phi) dy + \int_{\mathbb{R}^3} \hat{\Gamma} \Delta \psi u_\phi \cos(\theta - \phi) dy.
\end{aligned} \tag{3.27}$$

The cylindrical coordinate representation of Γ is denoted by $\hat{\Gamma} = \hat{\Gamma}(r, \rho, \theta - \phi, z - \kappa)$:

$$\hat{\Gamma} = \frac{1}{4\pi \sqrt{r^2 + \rho^2 - 2r\rho \cos(\theta - \phi) + (z - \kappa)^2}}.$$

Since \mathbf{e}_ρ and \mathbf{e}_r are different, they cause extra computations involving with $\cos(\phi - \theta)$. Direct computations yield that

$$\begin{aligned}
\frac{\partial \hat{\Gamma}}{\partial \rho} &= -\frac{1}{4\pi} \frac{\rho - r \cos(\theta - \phi)}{(r^2 + \rho^2 - 2r\rho \cos(\theta - \phi) + (z - \kappa)^2)^{\frac{3}{2}}}, \\
\frac{\partial \hat{\Gamma}}{\partial \kappa} &= \frac{1}{4\pi} \frac{(z - \kappa)}{(r^2 + \rho^2 - 2r\rho \cos(\theta - \phi) + (z - \kappa)^2)^{\frac{3}{2}}}.
\end{aligned}$$

Define $\Gamma_i = \hat{\Gamma}_i(r, \rho, z - \kappa)$, $i = 0, \dots, 5$, by

$$\begin{aligned}
\Gamma_0 &= \int_0^{2\pi} \hat{\Gamma}(r, \rho, \phi, z - \kappa) d\phi, & \Gamma_1 &= \int_0^{2\pi} \hat{\Gamma}(r, \rho, \phi, z - \kappa) \cos \phi d\phi, \\
\Gamma_2 &= \int_0^{2\pi} \frac{\partial \hat{\Gamma}}{\partial \rho}(r, \rho, \phi, z - \kappa) d\phi, & \Gamma_3 &= \int_0^{2\pi} \frac{\partial \hat{\Gamma}}{\partial \rho}(r, \rho, \phi, z - \kappa) \cos \phi d\phi, \\
\Gamma_4 &= \int_0^{2\pi} \frac{\partial \hat{\Gamma}}{\partial \kappa}(r, \rho, \phi, z - \kappa) d\phi, & \Gamma_5 &= \int_0^{2\pi} \frac{\partial \hat{\Gamma}}{\partial \kappa}(r, \rho, \phi, z - \kappa) \cos \phi d\phi.
\end{aligned}$$

Lemma 3.10. *We have the following integral representation of u_θ in terms of ω_r and ω_z :*

$$\begin{aligned}
u_\theta(r, z) &= - \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_5 \psi \omega_\rho \rho d\rho d\kappa + \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_3 \psi \omega_\kappa \rho d\rho d\kappa - \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \frac{\partial \psi}{\partial \kappa} \omega_\rho \rho d\rho d\kappa \\
&\quad + \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \frac{\partial \psi}{\partial \rho} \omega_\kappa \rho d\rho d\kappa + 2 \int_{-\infty}^{\infty} \int_0^{\infty} (\Gamma_3 \frac{\partial \psi}{\partial \rho} + \Gamma_5 \frac{\partial \psi}{\partial \kappa}) u_\phi \rho d\rho d\kappa \\
&\quad + \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \Delta \psi u_\phi \rho d\rho d\kappa.
\end{aligned} \tag{3.28}$$

To our purpose, we need the following estimates for Γ_i , $i = 1, \dots, 5$.

$$|\Gamma_i(r, \rho, z - \kappa)| \leq \frac{1}{\sqrt{(r - \rho)^2 + (z - \kappa)^2}} \quad \text{for } i = 0, 1, \tag{3.29}$$

$$|\Gamma_i(r, \rho, z - \kappa)| \leq \frac{\rho + r}{[(r - \rho)^2 + (z - \kappa)^2]^{\frac{3}{2}}} \quad \text{for } i = 2, 3, \tag{3.30}$$

$$|\Gamma_i(r, \rho, z - \kappa)| \leq \frac{|z - w|}{[(r - \rho)^2 + (z - \kappa)^2]^{\frac{3}{2}}} \quad \text{for } i = 4, 5. \tag{3.31}$$

For Γ_2, Γ_3 and Γ_5 , we have extra decay in r .

Lemma 3.11. Suppose $\frac{1}{4} \leq \frac{\rho}{r} \leq 16$, then

$$|\Gamma_2(r, \rho, z - \kappa)| \leq \frac{c}{\rho \sqrt{(\rho - r)^2 + (z - \kappa)^2}}, \quad (3.32)$$

$$|\Gamma_3(r, \rho, z - \kappa)| \leq \frac{c}{\rho \sqrt{(\rho - r)^2 + (z - \kappa)^2}}, \quad (3.33)$$

$$|\Gamma_5(r, \rho, z - \kappa)| \leq \frac{c}{r} \frac{|z - \kappa|}{(\rho - r)^2 + (z - \kappa)^2}. \quad (3.34)$$

Proof. (3.32) and (3.34) have been proved in Lemma 2.3 in [7]. It suffices to show (3.33). Since

$$\begin{aligned} \Gamma_3(r, \rho, z - \kappa) &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{\rho \cos \phi - r}{(r^2 + \rho^2 - 2r\rho \cos \phi + (z - \kappa)^2)^{\frac{3}{2}}} d\phi \\ &\quad - \frac{r}{4\pi} \int_0^{2\pi} \frac{\sin^2 \phi d\phi}{(r^2 + \rho^2 - 2r\rho \cos \phi + (z - \kappa)^2)^{\frac{3}{2}}} \\ &:= -\Gamma_2(\rho, r, z - \kappa) + rJ(r, \rho, z - \kappa), \\ |J(r, \rho, z - \kappa)| &\leq 2 \int_{-1}^1 \frac{dt}{[r^2 + \rho^2 - 2r\rho t + (z - \kappa)^2]^{\frac{3}{2}}} \\ &= \frac{2}{r\rho} \left[\frac{1}{\sqrt{(r - \rho)^2 + (z - \kappa)^2}} - \frac{1}{(r + \rho)^2 + (z - \kappa)^2} \right] \\ &\leq \frac{2}{r\rho \sqrt{(r - \rho)^2 + (z - \kappa)^2}}. \end{aligned}$$

Hence one has (3.33). □

Lemma 3.12. There is a positive constant $c(M)$ such that

$$|u_\theta(r, z)| \leq c(M) \left(\frac{\log r}{r} \right)^{\frac{1}{2}} \quad (3.35)$$

for large r , uniformly in z .

Proof. From Lemma 3.10. we have

$$\begin{aligned} u_\theta(r, z) &= \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_3 \psi \omega_\kappa \rho d\rho d\kappa - \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_5 \psi \omega_\rho \rho d\rho d\kappa + 2 \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_3 \frac{\partial \psi}{\partial \rho} u_\phi \rho d\rho d\kappa \\ &\quad + 2 \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_5 \frac{\partial \psi}{\partial \kappa} u_\phi \rho d\rho d\kappa + \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \frac{\partial \psi}{\partial \rho} \omega_\kappa \rho d\rho d\kappa - \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \frac{\partial \psi}{\partial \kappa} \omega_\rho \rho d\rho d\kappa \\ &\quad + \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \left(\partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi \right) u_\phi \rho d\rho d\kappa + \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma_1 \partial_\kappa^2 \psi u_\phi \rho d\rho d\kappa := \sum_{k=1}^8 T_k. \end{aligned}$$

(1) *Estimates of T_1, T_2 .* The estimates of T_1 and T_2 are similar. We define the integration domain D by

$$D = \{r/4 < \rho < 4r, -z_1 < \kappa < z_1\}.$$

We decompose the domain of D into two subregions. Let

$$\begin{aligned} A &= \{(\rho, \kappa) \in [r/4, 4r] \times [-z_1, z_1] : |r - \rho| \leq 1\}, \\ B &= \{(\rho, \kappa) \in [r/4, 4r] \times [-z_1, z_1] : |r - \rho| > 1\}. \end{aligned}$$

Then

$$\begin{aligned}
T_1 &\leq r \sup_{\substack{r/4 < \rho < 4r \\ -z_1 < \kappa < z_1}} |\omega_\kappa(\rho, \kappa)| \int_A |\Gamma_3(r, \rho, z - \kappa)| d\rho d\kappa \\
&\quad + \left(\int_{r/4}^{4r} \int_{-z_1}^{z_1} |\omega_\kappa(\rho, \kappa)|^2 \rho d\rho d\kappa \right)^{\frac{1}{2}} \left(\int_B |\Gamma_3(r, \rho, z - \kappa)|^2 \rho d\rho d\kappa \right)^{\frac{1}{2}}.
\end{aligned}$$

From Lemma 3.11, we have the following estimates

$$\begin{aligned}
&\int_A |\Gamma_3(r, \rho, z - \kappa)| d\rho d\kappa \leq \frac{c}{r} \int_0^1 \int_0^{z_1} \frac{ds dt}{\sqrt{s^2 + t^2}} \\
&\leq \frac{c}{r} \int_0^1 dt \left(\int_0^1 \frac{d\theta}{\sqrt{1 + \theta^2}} + \int_1^{\frac{z_1}{t}} \frac{d\theta}{\theta} \right) \\
&\leq \frac{c}{r} \left(c + \int_0^1 (\log z_1 - \log t) dt \right) \leq \frac{c + c \log z_1}{r}, \\
&\int_B |\Gamma_3(r, \rho, z - \kappa)|^2 d\rho d\kappa \leq \frac{c}{r} \int_1^{5r} \int_0^{z_1} \frac{ds dt}{t^2 + s^2} \\
&\leq \frac{c}{r} \int_1^{5r} \frac{dt}{t} \int_0^{z_1/t} \frac{d\theta}{1 + \theta^2} \leq \frac{c \log r}{r}.
\end{aligned}$$

Hence by (3.23), we have

$$|T_1| \leq \frac{c + c \log z_1}{r} + c \left(\frac{\log r}{r} \right)^{\frac{1}{2}}.$$

(2) *Estimates of T_3, T_5 and T_7 .* Since $|\frac{\partial \psi}{\partial \rho}| \leq \frac{c}{\rho}$,

$$\begin{aligned}
|T_3| &\leq \frac{c}{r} \|u_\phi\|_{L^6(\mathbb{R}^3)} \left(\int_{r/4}^{r/2} \int_{-\infty}^{\infty} |\Gamma_3|^{\frac{6}{5}} dx + \int_{2r}^{4r} \int_{-\infty}^{\infty} |\Gamma_3|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \\
&\leq \frac{c}{r} \left[\left(\int_{r/4}^{r/2} + \int_{2r}^{4r} \right) \rho d\rho \int_{-\infty}^{\infty} \frac{d\kappa}{r^{6/5} [(\rho - r)^2 + (z - \kappa)^2]^{\frac{3}{5}}} \right]^{\frac{6}{5}} \\
&\leq \frac{c}{r^{1/2}}.
\end{aligned}$$

Note that

$$\int_{-\infty}^{\infty} |\Gamma_1(r, \rho, z - \kappa)|^2 d\kappa \leq \frac{c}{|r - \rho|},$$

then we have

$$\begin{aligned}
|T_5| &\leq \frac{c}{r} \|\omega_\kappa\|_{L^2(\mathbb{R}^3)} \left[\left(\int_{r/4}^{r/2} + \int_{2r}^{4r} \right) \rho d\rho \int_{-\infty}^{\infty} |\Gamma_1|^2 d\kappa \right]^{\frac{1}{2}} \\
&\leq \frac{c}{r} \left[\left(\int_{r/4}^{r/2} + \int_{2r}^{4r} \right) \frac{\rho d\rho}{|r - \rho|} \right]^{1/2} \leq \frac{c}{r^{1/2}}, \\
|T_7| &\leq \frac{c}{r^2} \left(\int_{r/4}^{r/2} + \int_{2r}^{4r} \right) \rho d\rho \left(\int_{-\infty}^{\infty} |\Gamma_1|^2 d\kappa \right)^{1/2} \left(\int_{-\infty}^{\infty} |\omega_\rho|^2 d\kappa \right)^{1/2} \\
&\leq \frac{c}{r^2} \left(\int_{r/4}^{r/2} + \int_{2r}^{4r} \right) \frac{\rho d\rho}{\sqrt{|r - \rho|}} \leq \frac{c}{r^{1/2}}.
\end{aligned}$$

(3) *Estimates of T_4, T_6 and T_8 .* Since $|\frac{\partial \psi}{\partial w}| \leq \frac{c}{z_1}$ and $|\frac{\partial^2 \psi}{\partial w^2}| \leq \frac{c}{z_1^2}$, we observe that

$$\begin{aligned}
|T_4| &\leq \frac{c}{z_1} \|u_\phi\|_{L^6(\mathbb{R}^3)} \left(\int_{r/4}^{4r} \int_{-\infty}^{\infty} \frac{c}{r^{\frac{6}{5}}} \frac{|z - \kappa|^{\frac{6}{5}}}{[(r - \rho)^2 + (z - \kappa)^2]^{\frac{3}{5}}} d\kappa \right)^{\frac{5}{6}} \\
&\leq \frac{c}{rz_1} \left(\int_{r/4}^{4r} \frac{1}{|r - \rho|^{\frac{1}{3}}} \rho d\rho \right)^{\frac{5}{6}} \leq \frac{cr^{1/2}}{z_1}, \\
|T_6| &\leq \frac{c}{z_1} \int_{r/4}^{4r} \left(\int_{-\infty}^{\infty} |\Gamma_1|^2 d\kappa \right)^{1/2} \left(\int_{-\infty}^{\infty} |\omega_\rho|^2 d\kappa \right)^{1/2} \rho d\rho \\
&\leq \frac{c}{z_1} \int_{r/4}^{4r} \frac{1}{\sqrt{|r - \rho|}} \frac{1}{\sqrt{\rho}} \rho d\rho \leq \frac{cr}{z_1}, \\
|T_8| &\leq \frac{c}{z_1^2} \int_{r/4}^{4r} \left(\int_{-\infty}^{\infty} |\Gamma_1|^2 d\kappa \right)^{1/2} \left(\int_{-\infty}^{\infty} |u_\phi|^2 d\kappa \right)^{1/2} \rho d\rho \\
&\leq \frac{c}{z_1^2} \int_{r/4}^{4r} \frac{1}{\sqrt{|r - \rho|}} \rho d\rho \leq \frac{cr^{3/2}}{z_1^2}.
\end{aligned}$$

We choose $z_1 = r^2$, then combining all the above estimates, we finally obtain (3.35). \square

Proof of Theorem 1.1. We have proved (1.12) in Lemma 3.12. By (1.8) and (1.12), we see that (3.4) holds for $\delta = (\frac{1}{2})^-$, hence by (3.8), we have (1.13). Applying Lemma 3.5, we can take $\delta_1 = (\frac{1}{2})^-$ and $\delta_2 = 1^-$, hence by (3.15), we obtain (1.14). Then we use Lemma 3.8, where we can take $\delta = (\frac{1}{2})^-$, $\gamma = (\frac{1}{8})^-$, (3.22) implies (1.15). Since ∇u_θ can be expressed as singular integral operators of ω_r and ω_z , we can use A_p weight to derive same weighted energy estimates of ∇u_θ from (3.19)-(3.21). Then by Lemma 2.1, we obtain (1.16). \square

4 Proof of Theorem 1.2.

Proof of Theorem 1.2. Step 1. We have the following weighted estimates for $\Omega := \frac{\omega_\theta}{r}$:

$$\int_{-\infty}^{\infty} \int_1^{\infty} \sqrt{z^2 + 1} |\nabla \Omega(r, z)|^2 r dr dz < \infty, \quad (4.1)$$

$$\int_{-\infty}^{\infty} \int_1^{\infty} \sqrt{z^2 + 1} |\nabla \partial_z \Omega(r, z)|^2 r dr dz < \infty. \quad (4.2)$$

For the smooth axially symmetric flows with no swirl, Ω satisfies the following equation

$$(u_r \partial_r + u_z \partial_z) \Omega = (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \Omega + \frac{1}{r} (\partial_z f_r - \partial_r f_z). \quad (4.3)$$

Multiplying (4.3) by $(z^2 + 1)^{\frac{d_1}{2}} \eta^2 \Omega$ and integrating over \mathbb{R}^3 , then we get an integral identity with

left and right hand sides as

$$\begin{aligned}
LHS &= -2\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_1}{2}} \eta \eta' \frac{ru_r + zu_z}{\sqrt{r^2 + z^2}} \Omega^2 r dr dz \\
&\quad - \pi \int_{-\infty}^{\infty} \int_0^{\infty} d_1 (z^2 + 1)^{\frac{d_1}{2}-1} z \eta^2 u_z \Omega^2 r dr dz := E_1 + E_2, \\
RHS &= -2\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_1}{2}} \eta^2 |\nabla \Omega|^2 r dr dz - 4\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_1}{2}} \eta \eta' \frac{r \partial_r \Omega + z \partial_z \Omega}{\sqrt{r^2 + z^2}} \Omega r dr dz \\
&\quad - 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} d_1 (z^2 + 1)^{\frac{d_1}{2}-1} z \eta^2 \Omega \partial_z \Omega r dr dz - 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_1}{2}} 2\eta \eta' \frac{r}{\sqrt{r^2 + z^2}} \Omega^2 r dr dz \\
&\quad + \int_{\mathbb{R}^3} (z^2 + 1)^{\frac{d_1}{2}} \eta^2 \Omega \frac{1}{r} (\partial_z f_r - \partial_r f_z) dx := \sum_{k=1}^5 F_k.
\end{aligned}$$

Take $d_1 = 1$, since $|\sqrt{z^2 + 1} \phi'| \leq C$, then

$$\begin{aligned}
|E_1| &\leq C \|\Omega\|_{L^2(\mathbb{R}^3)}^2, \quad |E_2| \leq C \|u_z\|_{L^\infty(\mathbb{R}^3)} \|\Omega\|_{L^2(\mathbb{R}^3)}^2, \\
\sum_{k=2}^3 |F_k| &\leq C \|\Omega\|_{L^2(\mathbb{R}^3)} \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}, \\
|F_4| &\leq C \int_{-\infty}^{\infty} \int_1^{\infty} \Omega^2(r, z) dr dz \leq \|\Omega\|_{L^2(\mathbb{R}^3)}^2, \\
|F_5| &\leq C \|\Omega\|_{L^2(\mathbb{R}^3)} \|(|z| + 1) \frac{\partial_z f_r - \partial_r f_z}{r}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Finally, we obtain (4.1) by letting $\rho_0 \rightarrow \infty$.

To derive the estimate (4.2), we use the equation for $\partial_z \Omega$.

$$\partial_z \left((u_r \partial_r + u_z \partial_z) \Omega \right) = (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \partial_z \Omega + \frac{1}{r} \partial_z (\partial_z f_r - \partial_r f_z). \quad (4.4)$$

Multiplying (4.4) by $(z^2 + 1)^{\frac{d_2}{2}} \eta^2 \partial_z \Omega$ and integrating over \mathbb{R}^3 , then we get an integral identity with left and right hand sides as

$$\begin{aligned}
LHS &= - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \Omega (z^2 + 1)^{\frac{d_2}{2}} \eta^2 \partial_z^2 \Omega dx - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \Omega d_2 (z^2 + 1)^{\frac{d_2}{2}} \eta \eta' \frac{z}{\sqrt{r^2 + z^2}} \partial_z \Omega dx \\
&\quad - \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \Omega d_2 (z^2 + 1)^{\frac{d_2}{2}-1} z \eta^2 \partial_z \Omega dx := E'_1 + E'_2 + E'_3, \\
RHS &= -2\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_2}{2}} \eta^2 |\nabla \partial_z \Omega|^2 r dr dz - 4\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_2}{2}} \eta \eta' \frac{r \partial_{rz}^2 \Omega + z \partial_z^2 \Omega}{\sqrt{r^2 + z^2}} \partial_z \Omega r dr dz \\
&\quad - 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} d_2 (z^2 + 1)^{\frac{d_2}{2}-1} z \eta^2 \partial_z \Omega \partial_z^2 \Omega r dr dz - 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} (z^2 + 1)^{\frac{d_2}{2}} 2\eta \eta' \frac{r}{\sqrt{r^2 + z^2}} (\partial_z \Omega)^2 r dr dz \\
&\quad - \int_{\mathbb{R}^3} (z^2 + 1)^{\frac{d_2}{2}} \eta^2 \partial_z^2 \Omega \frac{1}{r} (\partial_z f_r - \partial_r f_z) dx - 2 \int_{\mathbb{R}^3} (z^2 + 1)^{\frac{d_2}{2}} \eta \eta' \frac{z}{\sqrt{r^2 + z^2}} \partial_z \Omega \frac{1}{r} (\partial_z f_r - \partial_r f_z) dx \\
&:= \sum_{k=1}^6 F'_k.
\end{aligned}$$

Then take $d_2 = 1$, we get

$$\begin{aligned}
|E'_1| &\leq \| (u_r, u_z) \|_{L^\infty} \| (z^2 + 1)^{\frac{1}{4}} \nabla \Omega \|_{L^2} \| \eta (z^2 + 1)^{\frac{1}{4}} |\nabla \partial_z \Omega| \|_{L^2}, \\
|E'_2| + |E'_3| &\leq \| (u_r, u_z) \|_{L^\infty} \| \nabla \Omega \|_{L^2}^2, \\
|F'_2| + |F'_3| &\leq C \| \nabla \Omega \|_{L^2} \| \nabla \partial_z \Omega \|_{L^2}, \\
|F'_4| &\leq \int_{-\infty}^{\infty} \int_1^{\infty} |\partial_z \Omega|^2 dr dz \leq \| \nabla \Omega \|_{L^2}^2, \\
|F'_5| &\leq \epsilon \| (|z|^2 + 1)^{\frac{1}{4}} \eta \nabla \partial_z \Omega \|_{L^2}^2 + C(\epsilon) \| (|z| + 1)^{\frac{1}{2}} \frac{\partial_z f_r - \partial_r f_z}{r} \|_{L^2(\mathbb{R}^3)}^2, \\
|F'_6| &\leq C \| \nabla \Omega \|_{L^2(\mathbb{R}^3)} \| \frac{\partial_z f_r - \partial_r f_z}{r} \|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Hence (4.2) is derived by letting $\rho_0 \rightarrow \infty$.

Step 2. Now we can derive the decay rate for ω_θ . Combining the results in Lemma 3.2 and (4.1)-(4.2), then

$$\int_{\mathbb{R}^3} r^2 |\Omega(r, z)|^2 dx < \infty, \quad (4.5)$$

$$\int_{\mathbb{R}^3} (r^{3+\delta} + |z|) |\nabla \Omega(r, z)|^2 dx < \infty, \quad (4.6)$$

$$\int_{\mathbb{R}^3} (r^{3+3\delta} + |z|) |\nabla \partial_z \Omega(r, z)|^2 dx < \infty, \quad (4.7)$$

where δ can be any constant less than $\frac{1}{2}$. Fix $d > 1$, then for each $n \in \mathbb{N}$,

$$\int_{2^n}^{2^{n+1}} \int_d^\infty r^2 |\Omega(r, z)|^2 r dr dz < \infty.$$

By mean value theorem, there exists $z_n \in [2^n, 2^{n+1}]$ such that

$$\int_d^\infty r^2 |\Omega(r, z_n)|^2 r dr \leq \frac{C}{z_n}.$$

Then for any z , choose $z_n > z$ and

$$\begin{aligned}
\int_d^\infty |\Omega(r, z)|^2 r dr &= \int_d^\infty |\Omega(r, z_n)|^2 r dr - 2 \int_d^\infty \int_z^{z_n} \Omega(r, t) \partial_t \Omega(r, t) r dr dt := I_1 + I_2, \\
|I_2| &\leq \left(\int_d^\infty \int_z^{z_n} |\Omega(r, t)|^2 r dr dt \right)^{1/2} \left(\int_d^\infty \int_z^{z_n} |\partial_t \Omega(r, t)|^2 r dr dt \right)^{1/2} \leq \frac{C}{d|z|^{1/2}}.
\end{aligned}$$

Letting $z_n \rightarrow \infty$, then $I_1 \rightarrow 0$ and

$$\int_d^\infty |\Omega(r, z)|^2 r dr \leq \frac{C}{d|z|^{1/2}}. \quad (4.8)$$

Similarly, one can find $z_n \in [2^n, 2^{n+1}]$ such that

$$\begin{aligned} \int_d^\infty |\nabla \Omega(r, z_n)|^2 r dr &\leq \frac{C}{z_n^2}, \\ \int_d^\infty |\nabla \Omega(r, z)|^2 r dr &= \int_d^\infty |\nabla \Omega(r, z_n)|^2 r dr - 2 \int_d^\infty \int_z^{z_n} \nabla \Omega(r, t) \cdot \partial_t \nabla \Omega(r, t) r dr dt := J_1 + J_2, \\ |J_2| &\leq \left(\int_d^\infty \int_z^{z_n} |\nabla \Omega(r, t)|^2 r dr dt \right)^{1/2} \left(\int_d^\infty \int_z^{z_n} |\partial_t \nabla \Omega(r, t)|^2 r dr dt \right)^{1/2} \\ &\leq \begin{cases} \frac{C}{d^{3+2\delta}}, \\ \frac{C}{|z|}. \end{cases} \end{aligned}$$

Letting $n \rightarrow \infty$, $J_1 \rightarrow 0$. Take $\delta = (\frac{1}{2})^-$ and $J_2 \leq \min\{\frac{C}{d^{3+2\delta}}, \frac{C}{|z|}\}$

$$\int_d^\infty |\nabla \Omega(r, z)|^2 r dr \leq \left(\frac{C}{d^{4-}}\right)^{\frac{1}{4}} \left(\frac{C}{|z|}\right)^{\frac{3}{4}} \leq \frac{C}{d|z|^{(\frac{3}{4})^-}}.$$

Finally,

$$\begin{aligned} |\Omega(d, z)|^2 &= \frac{1}{r_1 - d} \int_d^{r_1} |\Omega(r, z)|^2 dr + (|\Omega(r, z)|^2 - \frac{1}{r_1 - d} \int_d^{r_1} |\Omega(r, z)|^2 dr) := H_1 + H_2, \\ |H_2| &= \left| |\Omega(d, z)|^2 - |\Omega(d_*, z)|^2 \right| \leq 2 \int_d^{r_1} |\Omega(r, z) \partial_r \Omega(r, z)| dr \\ &\leq \frac{C}{d} \left(\int_d^\infty |\Omega(r, z)|^2 r dr \right)^{1/2} \left(\int_d^\infty |\nabla \Omega(r, z)|^2 r dr \right)^{1/2} \\ &\leq \frac{C}{d} \left(\frac{C}{d|z|} \right)^{1/2} \left(\frac{C}{d|z|^{(\frac{3}{4})^-}} \right)^{\frac{1}{2}} \leq \frac{C}{d^2 |z|^{(\frac{5}{8})^-}}, \end{aligned}$$

which implies that

$$|\omega(d, z)| \leq \frac{C}{|z|^{(\frac{5}{16})^-}}. \quad (4.9)$$

Together with Theorem 1.1, we have

$$|\omega(r, z)| \leq \frac{C}{\rho^{(\frac{5}{16})^-}}, \quad \forall (r, z) \in \mathbb{R}_+ \times \mathbb{R}, \rho = \sqrt{r^2 + z^2}. \quad (4.10)$$

Step 3. Now we derive the decay rate of \mathbf{u} . Fix any $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$, define a cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\psi(\mathbf{y}) \equiv 1$ for $\forall \mathbf{y} \in B_{\rho/4}(\mathbf{x})$ and $\psi(\mathbf{y}) \equiv 0$ for $\forall \mathbf{y} \notin B_{\rho/2}(\mathbf{x})$, where $\rho = |\mathbf{x}|$. One can require that $|\nabla \psi(\mathbf{y})| \leq \frac{C}{|\mathbf{y}|}$, $|\nabla^2 \psi(\mathbf{y})| \leq \frac{C}{|\mathbf{y}|^2}$ for $\forall \mathbf{y} \in D := B_{\rho/2}(\mathbf{x}) \setminus B_{\rho/4}(\mathbf{x})$. Setting $\mathbf{u}(\mathbf{x}) = u_r \mathbf{e}_r + u_z \mathbf{e}_z$, since $\text{curl } \mathbf{v} = \omega_\theta \mathbf{e}_\theta$, then

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &= - \int_{\mathbb{R}^3} \nabla_{\mathbf{y}} \Gamma(\mathbf{x}, \mathbf{y}) \times (\omega_\phi(\mathbf{y}) \psi(\mathbf{y}) \mathbf{e}_\phi) d\mathbf{y} - \int_{\mathbb{R}^3} \Gamma(\mathbf{x}, \mathbf{y}) ((\nabla_{\mathbf{y}} \psi(\mathbf{y}) \times \mathbf{e}_\phi)) \omega_\phi(\mathbf{y}) d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^3} \Gamma(\mathbf{x}, \mathbf{y}) (\Delta_{\mathbf{y}} \psi)(\mathbf{y}) \mathbf{v}(\mathbf{y}) d\mathbf{y} + 2 \int_{\mathbb{R}^3} (\nabla_{\mathbf{y}} \Gamma)(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} \psi)(\mathbf{y}) \mathbf{v}(\mathbf{y}) d\mathbf{y} \\ &:= K_1 + K_2 + K_3 + K_4. \end{aligned} \quad (4.11)$$

We estimate $K_i, i = 2, 3, 4$ as follows.

$$\begin{aligned} |K_2| &\leq \frac{C}{\rho} \left(\int_D |\Gamma(\mathbf{x} - \mathbf{y})|^2 d\mathbf{y} \right)^{1/2} \left(\int_D |\omega_\phi(\mathbf{y})|^2 d\mathbf{y} \right)^{1/2} \leq \frac{C}{\rho^{1/2}}, \\ |K_3| &\leq \frac{C}{\rho^2} \left(\int_D |\Gamma(\mathbf{x} - \mathbf{y})|^{\frac{6}{5}} d\mathbf{y} \right)^{\frac{5}{6}} \left(\int_D |\mathbf{v}(\mathbf{y})|^6 d\mathbf{y} \right)^{\frac{1}{6}} \leq \frac{C}{\rho^{1/2}}, \\ |K_4| &\leq \frac{C}{\rho} \left(\int_D |\nabla \Gamma(\mathbf{x} - \mathbf{y})|^{\frac{6}{5}} d\mathbf{y} \right)^{\frac{5}{6}} \left(\int_D |\mathbf{v}(\mathbf{y})|^6 d\mathbf{y} \right)^{\frac{1}{6}} \leq \frac{C}{\rho^{1/2}}. \end{aligned}$$

For the estimate of K_1 , fix a $d \in (0, \frac{\rho}{2})$, which will be determined later, then

$$\begin{aligned} |K_1| &\leq \sup_{\mathbf{y} \in B_d(\mathbf{x})} |\omega_\phi(\mathbf{y})| \int_{B_d(\mathbf{x})} |\nabla \Gamma(\mathbf{x} - \mathbf{y})| d\mathbf{y} \\ &\quad + \left(\int_{B_{\rho/2}(\mathbf{x}) \setminus B_d(\mathbf{x})} |\nabla \Gamma(\mathbf{x} - \mathbf{y})|^2 d\mathbf{y} \right)^{\frac{1}{2}} \left(\int_{B_{\rho/2}(\mathbf{x}) \setminus B_d(\mathbf{x})} |\omega_\phi(\mathbf{y})|^2 d\mathbf{y} \right)^{\frac{1}{2}} \\ &\leq C \rho^{-(\frac{5}{16})^-} d + C d^{-\frac{1}{2}}. \end{aligned}$$

By choosing $d = \rho^{(\frac{5}{24})^-}$, we obtain the optimal bound for $|K_1| \leq \frac{C}{\rho^{(\frac{5}{48})^-}}$.

□

Remark 4.1. Since there are no improved decay estimates of $\Gamma_2, \Gamma_3, \Gamma_5$ in w in (3.32)-(3.33), it seems difficult to use the argument in Lemma 3.12 to get better decay estimates of u_θ . So we use (4.11) directly. From the estimates of K_2, K_3 and K_4 , one may also see the difficulties to improve the decay rates in Theorem 1.1.

Remark 4.2. The decay rates in Theorem 1.2 are not optimal. However, the estimate of F_4 prevents us from improving weighted estimates in the O_z direction.

Remark 4.3. One can also extend Theorem 1.2 to the exterior domain case.

Acknowledgement. Weng's research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2014047764). The author would like to thank Prof. Dongho Chae and Prof. Zhouping Xin for the stimulating discussions and constant encouragement and supports.

References

- [1] K. I. Babenko. *On the stationary solutions of the problem of flow past a body of a viscous incompressible fluid.* Math. Sbornik 91 (133) No. 1 (1973); English Transl.: Math. USSR Sbornik 20 1 (1973), 1-25.
- [2] D. Chae, J. Lee. *On the regularity of the axisymmetric solutions of the Navier-Stokes equations.* Math. Z. 239 (2002), no. 4, 645–671.
- [3] D. Chae. *Liouville-Type Theorem for the Forced Euler Equations and the Navier-Stokes Equations.* Commun. Math. Phys. 326: 37-48 (2014).
- [4] D. Chae, T. Yoneda. *On the Liouville theorem for the stationary Navier-Stokes equations in a critical space.* J. Math. Anal. Appl. 405 (2013), no. 2, 706–710.

- [5] D. Chae, S. Weng. *Liouville type theorems for the steady axially symmetric Navier-Stokes and magnetohydrodynamic equations*. Submitted, August 2015.
- [6] P. Deuring and G. P. Galdi. *On the asymptotic behavior of physically reasonable solutions to the stationary Navier-Stokes system in three-dimensional exterior domains with zero velocity at infinity*. J. Math. Fluid Mech. 2, no. 4 (2000), 353-364.
- [7] H. Choe, B. Jin. *Asymptotic properties of axi-symmetric D-solutions of the Navier-Stokes equations*. J. Math. Fluid. Mech. 11 (2009), 208-232.
- [8] R. Farwig. *The stationary Navier-Stokes equations in a 3D exterior domain*, in: *Recent topics on mathematical theory of viscous incompressible fluid*, 53-115, Lecture Notes Numer. Appl. Anal. 16, Kinokuniya, Tokyo, 1998.
- [9] R. Farwig and H. Sohr. *Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains*, in: *Theory of the Navier-Stokes equations*, 11-30, Ser. Adv. Math. Appl. Sci. 47, World Sci. Publ., River Edge, NJ, 1998.
- [10] R. Finn. *On Steady-State Solutions of the Navier-Stokes Partial Differential Equations*. Arch. Rational Mech. Anal., Vol. 3, 1959, 381-396.
- [11] R. Finn. *On the steady-state solutions of the Navier-Stokes equations. III*, Acta Math. 105 (1961), 197-244.
- [12] R. Finn. *On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems*. Arch. Rational Mech. Anal. 19 (1965), 363-406.
- [13] H. Fujita. *On the existence and regularity of the steady-state solutions of the Navier-Stokes theorem*. J. Fac. Sci. Univ. Tokyo Sect. I 9(1961), 59-102.
- [14] Giovanni P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. In: Steady State problems*, Springer Monographs in Mathematics, Second edition, 2011.
- [15] Giovanni P. Galdi. *On the Asymptotic Properties of Leray's Solutions to the Exterior Stationary Three-Dimensional Navier-Stokes Equations with Zero Velocity at Infinity*. Degenerate Diffusions, IMA Volumes in Mathematics and Its Applications Vol 47, Ni, W-M., Peletier, L. A. and Vasquez, J. L., Eds. Springer-Verlag, 95-103.
- [16] D. Gilbarg, H. F. Weinberger. *Asymptotic properties of Leray's solutions of the stationary two-dimensional Navier-Stokes equations*. Uspehi Mat. Nauk 29 (1974), no. 2 (176), 109C122. English transl.: Russian Math. Surveys 29, No. 2 (1974), 109-123.
- [17] D. Gilbarg, H. F. Weinberger. *Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral*. Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) 5 (1978), no. 2, 381-404.
- [18] A. Korolev, V. Sverak. *On the large-distance asymptotics of steady state solutions of the Navier-Stokes equations in 3D exterior domains*. Ann. I. H. Poincaré- AN 28 (2011) 303-313.
- [19] M. Korobkov, K. Pileckas and R. Russo. *The Liouville Theorem for the Steady-State Navier-Stokes Problem for Axially Symmetric 3D Solutions in Absence of Swirl*. J. Math. Fluid Mech. 17 (2015), 287-293.

- [20] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible fluid*. Gordon and Breach, 1969.
- [21] J. Leray. *Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique*. J. Math. Pures Appl. 12 (1933), 1-82.
- [22] S. A. Nazarov and K. I. Pileckas. *On steady Stokes and Navier-Stokes problems with zero velocity at infinity in a three-dimensional exterior domain*. Kyoto Univ. Math. J. 40, no. 3 (2000), 475-492.
- [23] A. Novotny, M. Padula. *Note on decay of solutions of steady Navier-Stokes equations in 3-D exterior domains*. Differential and Integral Equations, Vol. 8, no. 7, 1995, 1833-1842.
- [24] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton University Press, Princeton, 1993.